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# A BILINEAR FORM FOR SPIN MANIFOLDS

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ABSTRACT. This paper studies the bilinear form on $H^j(M; Z_2)$ defined by $[x, y] = x\,\mathrm{Sq}^2 y[M]$ when $M$ is a closed Spin manifold of dimension $2j + 2$. In analogy with the work of Lusztig, Milnor, and Peterson for oriented manifolds, the rank of this form on integral classes gives rise to a cobordism invariant.

**1. Introduction.** If $M^{2j+2}$ is a closed Spin manifold of dimension $2j + 2$ one has a symmetric bilinear form

$$[\ ,\ ]\colon H^j(M; Z_2) \otimes H^j(M; Z_2) \to Z_2\colon [x, y] = x\,\mathrm{Sq}^2 y[M].$$

To see that this form is symmetric, one uses the identity

$$\begin{aligned}(x\,\mathrm{Sq}^2 y + y\,\mathrm{Sq}^2 x)[M] &= (\mathrm{Sq}^2(xy) + \mathrm{Sq}^1 x\,\mathrm{Sq}^1 y)[M]\\ &= (v_2 xy + v_1 x\,\mathrm{Sq}^1 y)[M] = 0\end{aligned}$$

where $v_i$ denotes the $i$th Wu class of $M$ and $v_1 = 0 = v_2$ for Spin manifolds.

The main result of this paper is

PROPOSITION 1.1. *For a closed Spin manifold $M^{8k+2}$ of dimension $8k + 2$ and class $z \in H^{4k}(M; Z)$*

$$\rho z\,\mathrm{Sq}^2 \rho z[M] = \rho z\,\mathrm{Sq}^2 v_{4k}[M]$$

*where $\rho$ is the* mod 2 *reduction and $v_{4k}$ is the $4k$th Wu class of $M$.*

This result arose in answering a question of Edward Witten, who wished to know the structure of $\Omega_{11}^{\mathrm{Spin}}(K(Z, 4))$. In the process this formula was seen to hold for ten dimensional manifolds.

Considering [ , ] as defining a form on integral cohomology via $\rho$, one then has

COROLLARY 1.2. *For a closed Spin manifold $M^{8k+2}$ of dimension $8k + 2$*

$$w_4 w_{8k-2}[M] = v_{4k}\,\mathrm{Sq}^2 v_{4k}[M]$$

*is the rank modulo* 2 *of the form* [ , ] *on integral cohomology.*

*Note.* Here, the rank of the form is the dimension as $Z_2$ vector space of $H^{4k}(M; Z)$ modulo the annihilator of the form.

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Of course, these results are completely analogous to the work of Lusztig, Milnor, and Peterson [**LMP**], or originally Browder [**B1**], on the form $(x, y) = x\,\mathrm{Sq}^1 y[M]$ for oriented manifolds of dimension $4k+1$. The proofs are, unfortunately, rather more complicated, and involve the calculation of the Spin bordism of Eilenberg-Mac Lane spaces just outside the stable range. As a sidelight, this work helps to explain the work of Wilson [**W**] on the vanishing of Stiefel-Whitney classes in Spin manifolds. Knowledge of the form gives

COROLLARY 1.3. *For a closed Spin manifold $M^{8k+2}$ of dimension $8k+2$, the Stiefel-Whitney class $\mathrm{Sq}^3 v_{4k}$ is zero.*

In §2, the proof is begun by showing that there is a class $\theta \in H^*(B\mathrm{Spin}; Z_2)$ for which $\rho z\,\mathrm{Sq}^2 \rho z[M] = \rho z \cdot \tau^*(\theta)[M]$. In §3, the elementary properties of $\theta$ are described, and in the following section, $\theta$ is shown to be unique by a nasty calculation. §5 then collects the main results, and the final section contains an extension to mod 4 cohomology suggested by Steven Kahn.

The authors are indebted to Edward Witten, whose questions about the Spin bordism of Eilenberg-Mac Lane spaces led to this work; to Steven Kahn, whose suggestions led to an extension of the results; and to the National Science Foundation for financial support during this work.

**2. Spin bordism of Eilenberg-Mac Lane spaces.** The basic tool for analyzing the form [ , ] is given by

LEMMA 2.1. *There are exact sequences*

$$\cdots \to \pi_{r+i+1}(M\mathrm{Spin} \wedge X_r) \to \tilde{\Omega}^{\mathrm{Spin}}_{i+j}(K(Z, j)) \to H_i(B\mathrm{Spin}; Z) \to \pi_{r+i}(M\mathrm{Spin} \wedge X_r) \to \cdots,$$

$$\cdots \to \pi_{r+i+1}(M\mathrm{Spin} \wedge Y_r) \to \tilde{\Omega}^{\mathrm{Spin}}_{i+j}(K(Z_2, j)) \to H_i(B\mathrm{Spin}; Z_2) \to \pi_{r+i}(M\mathrm{Spin} \wedge Y_r) \to \cdots$$

*for r large.*

PROOF. One considers the cofibration $\Sigma^{r-j}K(\pi, j) \xrightarrow{g} K(\pi, r) \to W_r$ with $\pi = Z$ or $Z_2$ ($W_r = X_r$ or $Y_r$, respectively) and $r$ large, and applies reduced Spin bordism. Of course, $\tilde{\Omega}^{\mathrm{Spin}}_{r+i}(W_r) = \pi_{r+i}(M\mathrm{Spin} \wedge W_r)$ by interpreting Spin bordism as the homotopy of a spectrum, and $\tilde{\Omega}^{\mathrm{Spin}}_{r+i}(\Sigma^{r-j}K(\pi, j)) = \tilde{\Omega}^{\mathrm{Spin}}_{i+j}(K(\pi, j))$ using the suspension isomorphism. Finally, for $r$ and $s$ large

$$\tilde{\Omega}^{\mathrm{Spin}}_{r+i}(K(\pi, r)) = \pi_{8s+r+i}(M\mathrm{Spin}_{8s} \wedge K(\pi, r)) = \tilde{H}_{8s+i}(M\mathrm{Spin}_{8s}; \pi) = H_i(B\mathrm{Spin}_{8s}; \pi) = H_i(B\mathrm{Spin}; \pi).$$

Here $g$ is intended to be the map for which $g^* i_r = \sigma^{r-j} i_j$ with $\sigma$ denoting suspension and $i_r \in H^r(K(\pi, r); \pi)$ being the fundamental class. □

In order to analyze the 2-primary part of $\pi_{r+i}(M\mathrm{Spin} \wedge W_r)$, $W_r = X_r$ or $Y_r$, one uses mod 2 cohomology. Heavy use will be made of the structure of $M\mathrm{Spin}$, as described by Anderson, Brown, and Peterson [**ABP**]. In particular,

$$\tilde{H}^*(M\mathrm{Spin}; Z_2) \cong (\mathscr{A}/\mathscr{A}\,\mathrm{Sq}^1 + \mathscr{A}\,\mathrm{Sq}^2)U + (\mathscr{A}/\mathscr{A}\,\mathrm{Sq}^1 + \mathscr{A}\,\mathrm{Sq}^2)w_4^2 U$$

plus terms of higher dimension, where $U$ is the Thom class, $w_4 \in H^*(B\,\mathrm{Spin};\ Z_2)$ is the universal Stiefel-Whitney class, and $\mathscr{A}$ denotes the mod 2 Steenrod algebra. One then needs to know the structure of $\tilde{H}^*(W_r;\ Z_2)$ as a module over $\mathscr{A}_1$, the subalgebra of $\mathscr{A}$ generated by $\mathrm{Sq}^1$ and $\mathrm{Sq}^2$, which is

| 1 | $\mathrm{Sq}^1$ | $\mathrm{Sq}^2$ | $\mathrm{Sq}^3, \mathrm{Sq}^2\mathrm{Sq}^1$ | $\mathrm{Sq}^3\mathrm{Sq}^1$ | $\mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1$ | $\mathrm{Sq}^5\mathrm{Sq}^1$ |
|---|---|---|---|---|---|---|
| dim 0 | dim 1 | dim 2 | dim 3 | dim 4 | dim 5 | dim 6 |

From the cofibration

$$\Sigma^{r-j}K(\pi, j) \xrightarrow{g} K(\pi, r) \xrightarrow{h} W_r$$

one has an exact sequence

$$\begin{tikzcd} \tilde{H}^*(\Sigma^{r-j}K(\pi, j);\ Z_2) & \tilde{H}^*(K(\pi, r);\ Z_2) \arrow[l, "g^*"'] & \tilde{H}^*(W_r;\ Z_2) \arrow[l, "h^*"'] \arrow[from=1-1, to=1-3, bend right, "\delta"'] \end{tikzcd}$$

Within the stable range, $\tilde{H}^*(K(\pi, r);\ Z_2)$ has a basis given by the classes $\mathrm{Sq}^K i_r = \mathrm{Sq}^{k_1} \cdots \mathrm{Sq}^{k_s} i_r$ where $K = (k_1, \ldots, k_s)$ is an admissible sequence ($k_i \geqslant 2k_{i+1}$) and, for $\pi = Z$, $k_s > 1$. One knows that $H^*(K(\pi, j);\ Z_2)$ is the polynomial ring over $Z_2$ on the classes $\mathrm{Sq}^K i_j$ where $K$ is admissible, has $k_s > 1$ if $\pi = Z$, and has excess $e(K) = (k_1 - 2k_2) + (k_2 - 2k_3) + \cdots + (k_{s-1} - 2k_s) + k_s$ less than $j$. If $e(K) = j$, $\mathrm{Sq}^K i_j = (\mathrm{Sq}^{K'} i_j)^{2^t}$ for some $K'$ and $t$. Since

$$g^*(\mathrm{Sq}^K i_r) = \mathrm{Sq}^K g^*(i_r) = \mathrm{Sq}^K \sigma^{r-j} i_j = \sigma^{r-j}\mathrm{Sq}^K i_j,$$

one can readily analyze the kernel and cokernel of $g^*$. The kernel of $g^*$ has a basis given by the classes $\mathrm{Sq}^K i_r$ with $e(K) > j$, and the cokernel of $g^*$ is given by classes $\sigma^{r-j}(\mathrm{Sq}^{k_1} i_j \cdots \mathrm{Sq}^{k_t} i_j)$ with $t > 1$, modulo the classes $\sigma^{r-j}((\mathrm{Sq}^{K'} i_j)^{2^t})$.

As a special case, one can then consider $\pi = Z$, $j = 4k$, and write down $\tilde{H}^*(X_r;\ Z_2)$ in low dimensions. There is a basis given by

$$\begin{array}{ll} \dim(r + 4k + 1) & \{\mathrm{Sq}^{4k+1} i_r\}, \\ \dim(r + 4k + 2) & \{\mathrm{Sq}^{4k+2} i_r\}, \\ \dim(r + 4k + 3) & \{\mathrm{Sq}^{4k+3} i_r\},\ \delta\sigma^{r-4k} i_{4k}\mathrm{Sq}^2 i_{4k}, \\ \dim(r + 4k + 4) & \{\mathrm{Sq}^{4k+4} i_r\},\ \{\mathrm{Sq}^{4k+3}\mathrm{Sq}^2 i_r\},\ \delta\sigma^{r-4k} i_{4k}\mathrm{Sq}^3 i_{4k} \end{array}$$

and terms of higher degree. Here $\{x\}$ denotes a class mapping by $h^*$ to $x$, i.e. $h^*(\{x\}) = x$.

Being interested in the action of $\mathscr{A}_1$, one needs the Adem relations

$$\mathrm{Sq}^1\mathrm{Sq}^b = \begin{cases} \mathrm{Sq}^{b+1}, & b \text{ even } > 0, \\ 0, & b \text{ odd}, \end{cases}$$

and

$$\mathrm{Sq}^2\mathrm{Sq}^b = \begin{cases} \mathrm{Sq}^{b+2} + \mathrm{Sq}^{b+1}\mathrm{Sq}^1, & b \equiv 0, 3 \bmod 4, \\ \mathrm{Sq}^{b+1}\mathrm{Sq}^1, & b \equiv 1, 2 \bmod 4, \end{cases} \quad (b > 1).$$

Then one has $\mathrm{Sq}^1\{\mathrm{Sq}^{4k+1}i_r\} = 0$, $\mathrm{Sq}^1\{\mathrm{Sq}^{4k+2}i_r\} = \{\mathrm{Sq}^{4k+3}i_r\}$, i.e., $\mathrm{Sq}^1\{\mathrm{Sq}^{4k+2}i_r\}$ is a class which maps to $\mathrm{Sq}^{4k+3}i_r$ and $\{\mathrm{Sq}^{4k+3}i_r\}$ may be *chosen* to be $\mathrm{Sq}^1$ on the lower class, and

$$\mathrm{Sq}^1\,\delta\sigma^{r-4k}i_{4k}\,\mathrm{Sq}^2 i_{4k} = \delta\sigma^{r-4k}i_{4k}\,\mathrm{Sq}^3 i_{4k}.$$

Also, $\mathrm{Sq}^2\,\mathrm{Sq}^{4k+1}i_r = 0$, so there is a $\mu \in Z_2$ for which

$$\mathrm{Sq}^2\{\mathrm{Sq}^{4k+1}i_r\} = \mu\delta\sigma^{r-4k}i_{4k}\mathrm{Sq}^2 i_{4k}.$$

*Claim.* $\mu \neq 0$. To verify this, one may consider the effect of the assumption that $\mu = 0$. To begin, one notices that rationally $\tilde{\Omega}_{8k}^{\mathrm{Spin}}(K(Z,4k))$ has a nonzero class detected by $i_{4k}^2$ which goes to zero in $\tilde{\Omega}_{r+4k}^{\mathrm{Spin}}(K(Z,r))$, and so $\pi_{r+4k+1}(M\mathrm{Spin} \wedge X_r) = Z +$ torsion. One may then find a map

$$F \to M\mathrm{Spin}_{8s} \wedge X_r \xrightarrow{a} K(Z, 8s+r+4k+1) \times K(Z_2, 8s+r+4k+2)$$

with $F$ being the fiber, so that

$$a^*(i_{8s+r+4k+1}) = U\cdot\{\mathrm{Sq}^{4k+1}i_r\}, \quad a^*(i_{8s+r+4k+2}) = U\cdot\{\mathrm{Sq}^{4k+2}i_r\}.$$

There must then be a class $b \in H^{8s+r+4k+2}(F, Z_2)$ transgressing to kill $\mathrm{Sq}^2 i_{8s+r+4k+1}$, with $\mathrm{Sq}^1 b$ transgressing to $\mathrm{Sq}^3 i_{8s+r+4k+1}$. Thus $\pi_{8s+r+4k+2}(F) \cong Z_2$ and

$$\pi_{r+1}(M\mathrm{Spin} \wedge X_r) = \begin{cases} Z, & i = 4k+1, \\ \text{order } 4, & i = 4k+2, \end{cases}$$

modulo odd torsion.

If one now considers the case $k = 1$, one has the exact sequence

$$\begin{array}{ccccc}
\tilde{\Omega}_{10}^{\mathrm{Spin}}(K(Z,4)) & \xrightarrow{b} & H_6(B\mathrm{Spin};\, Z) & \to & \pi_{r+6}(M\mathrm{Spin} \wedge X_r) \\
\| & & \| & & \\
Z_2 + Z_2 & & Z_2 & & \\
 & \to & \tilde{\Omega}_9^{\mathrm{Spin}}(K(Z,4)) & \to & H_5(B\mathrm{Spin};\, Z) \\
 & & \| & & \| \\
 & & Z_2 & & 0
\end{array}$$

in which the groups $\tilde{\Omega}_*^{\mathrm{Spin}}(K(Z,4))$ are known from [**S**]. Here $b$ is epic; there is a closed Spin manifold $M^{10}$ and integral class $z \in H^4(M;\, Z)$ reducing to $w_4$ for which $w_6\rho z[M] = w_6 w_4[M] \neq 0$. (*Note.* A specific example of such a manifold is given in [**F**, p. 218].) Thus $\pi_{r+6}(M\mathrm{Spin} \wedge X_r) = Z_2$, and so $\mu = 1$ when $k = 1$.

One then has a commutative diagram

$$\begin{array}{ccccc}
HP^\infty \wedge \Sigma^{r-4k}K(Z,4) & \to & HP^\infty \wedge K(Z, r-4k+4) & \to & HP^\infty \wedge X_r \quad (k'=1) \\
\downarrow \Sigma c & & \downarrow c & & \downarrow d \\
\Sigma^{r-4k}K(Z,4k) & \to & K(Z,r) & \xrightarrow{e} & X_r
\end{array}$$

in which $c^*(i_r) = u^{k-1}i_{r-4k+4}$, $u \in H^4(HP^\infty;\, Z) = Z$ being a generator, with $\Sigma c$ being obtained by suspending the similar map, and with $d$ being the induced map on cofibers.

$$\begin{aligned} c^*e^*\{\mathrm{Sq}^{4k+1}i_r\} &= \mathrm{Sq}^{4k+1}c^*(i_r) \\ &= u^{2k-2}\mathrm{Sq}^5 i_{r-4k+4} + \text{terms with smaller powers of } u, \end{aligned}$$

so

$$d^*\{\mathrm{Sq}^{4k+1}i_{r-4k+4}\} = u^{2k-2}\{\mathrm{Sq}^5 i_{r-4k+4}\}$$
$$+ \text{ terms with smaller powers of } u.$$

Since $\mathrm{Sq}^2 u = 0 = \mathrm{Sq}^1 u$, this gives

$$d^*\left(\mathrm{Sq}^2\{\mathrm{Sq}^{4k+1}i_r\}\right) = u^{2k-2}\mathrm{Sq}^2\{\mathrm{Sq}^5 i_{r-4k+4}\}$$
$$+ \text{ terms with smaller powers of } u.$$

Thus $\mathrm{Sq}^2\{\mathrm{Sq}^{4k+1} i_r\} \neq 0$, and hence $\mu \neq 0$ for all $k$, completing the proof of the claim.

LEMMA 2.2. *There is a class $\theta \in H^{4k+2}(B\mathrm{Spin};\, Z_2)$ for which*

$$\rho z\, \mathrm{Sq}^2 \rho z[M] = \tau^*(\theta)\rho z[M]$$

*for all* Spin $M^{8k+2}$ *and $z \in H^{4k}(M;\, Z)$, where $\tau\colon M \to B\mathrm{Spin}$ classifies the tangent bundle.*

PROOF. Consider the diagram

$$\begin{array}{ccccc} \pi_{r+4k+3}(M\mathrm{Spin} \wedge X_r) & \xrightarrow{\partial} & \tilde{\Omega}^{\mathrm{Spin}}_{8k+2}(K(Z,4k)) & \to & H_{4k+2}(B\mathrm{Spin};\, Z) \\ & & \downarrow \phi & & \\ & & Z_2 & & \end{array}$$

where $\phi$ assigns to $f\colon\ M^{8k+2} \to K(Z,4k)$ the characteristic number $f^*(i_{4k})\cdot \mathrm{Sq}^2 f^*(i_{4k})[M^{8k+2}]$. Then $\phi \circ \partial(\alpha)$ is the value on $\alpha$ of the characteristic number $U \cdot \delta\sigma^{r-4k} i_{4k}\, \mathrm{Sq}^2 i_{4k} = \mathrm{Sq}^2(U \cdot \{\mathrm{Sq}^{4k+1} i_r\})$ and cohomology classes of this form vanish on homotopy ($\mathrm{Sq}^i$ is zero in a sphere), so $\phi$ is zero on the image of $\partial$.

Now $H_{4k+2}(B\mathrm{Spin};\, Z)$ is a $Z_2$ vector space and sits inside $H_{4k+2}(B\mathrm{Spin};\, Z_2)$, so there is a homomorphism $\psi\colon\ H_{4k+2}(B\mathrm{Spin};\, Z_2) \to Z_2$ or equivalently class $\theta \in H^{4k+2}(B\mathrm{Spin};\, Z_2)$ for which $\psi$ restricts to $\phi$ on the image of $\tilde{\Omega}^{\mathrm{Spin}}_{8k+2}(K(Z,4k))$. Now for $z \in H^{4k}(M;\, Z)$, $\psi(\tau_*([M] \cap \rho z)) = \tau^*(\theta)\rho z[M]$ then gives $\phi$ on the class of $(M, z)$, i.e., $\rho z \mathrm{Sq}^2 \rho z[M]$. □

Notice that the proof of the proposition has now been reduced to the identification of the class $\theta$. This will require more work.

**3. Describing $\theta$.** From the previous section one knows that there is a class $\theta$ in $H^{4k+2}(B\mathrm{Spin};\, Z_2)$ so that $\tau^*(\theta)\rho z[M] = \rho z \mathrm{Sq}^2 \rho z[M]$ for all $M$ and $z$. One now wishes to find this class.

LEMMA 3.1. *The class $\theta$ is only well defined in*

$$H^{4k+2}(B\mathrm{Spin};\, Z_2)/\mathrm{Sq}^1 H^{4k+1}(B\mathrm{Spin};\, Z_2).$$

PROOF. For $\eta \in H^{4k+1}(B\mathrm{Spin};\, Z_2)$,

$$\begin{aligned} \tau^*(\theta + \mathrm{Sq}^1\eta)\rho z[M] &= \tau^*(\theta)\rho z[M] + \left(\mathrm{Sq}^1\tau^*(\eta)\right)\cdot \rho z[M] \\ &= \rho z \mathrm{Sq}^2 \rho z[M] + \left(v_1 \tau^*(\eta)\rho z + \tau^*(\eta)\, \mathrm{Sq}^1 \rho z\right)[M] \\ &= \rho z \mathrm{Sq}^2 \rho z[M]. \end{aligned}$$

Thus, the class $\theta + \mathrm{Sq}^1\eta$ has the same property as does $\theta$. □

*Note*. This corresponds to the fact that $\tilde{\Omega}^{\mathrm{Spin}}_{8k+2}(K(Z,4k))$ maps into

$$H_{4k+2}(B\,\mathrm{Spin};\ Z) \subset H_{4k+2}(B\,\mathrm{Spin};\ Z_2),$$

with the classes in the image of $\mathrm{Sq}^1$ vanishing on integral homology.

LEMMA 3.2. *$\theta$ is nonzero in $H^{4k+2}(B\,\mathrm{Spin};\ Z_2)/\mathrm{Sq}^1\, H^{4k+1}(B\,\mathrm{Spin};\ Z_2)$.*

PROOF. It is sufficient to exhibit a manifold $M^{8k+2}$ and integral class $z \in H^{4k}(M;\ Z)$ for which $\rho z \mathrm{Sq}^2 \rho z[M] \neq 0$. For this one lets $M \subset \mathbf{C}P^2 \times \mathbf{C}P^{4k}$ be the Milnor hypersurface dual to $\alpha + \beta$, $\alpha \in H^2(\mathbf{C}P^2;\ Z)$ and $\beta \in H^2(\mathbf{C}P^{4k};\ Z)$ being the generators, and lets $z = \alpha\beta^{2k-1}$, or more precisely, the pullback to $M$. This is a Spin manifold, and the desired number is nonzero. $\square$

LEMMA 3.3. *$\mathrm{Sq}^1\theta \in H^{4k+3}(B\,\mathrm{Spin};\ Z_2)$ is a nonzero class with $\tau^*(\mathrm{Sq}^1\theta) = 0$ in the cohomology of every closed Spin manifold of dimension $8k+2$. Further, $\theta \in H^{4k+2}(B\,\mathrm{Spin};\ Z_2)/\mathrm{Sq}^1 H^{4k+1}(B\,\mathrm{Spin};\ Z_2)$ is determined by $\mathrm{Sq}^1\theta$.*

PROOF. According to Anderson, Brown, and Peterson [**ABP**, Proposition 6.1] $H(H^*(B\,\mathrm{Spin};\ Z_2), \mathrm{Sq}^1) = Z_2[1 \cdot \mathrm{Sq}^{2^i}, P_j]$ with $i \geqslant 2$, $j \neq 2^k$, is a polynomial ring on generators of dimensions divisible by 4, so $\mathrm{Sq}^1$ maps

$$H^{4k+2}(B\,\mathrm{Spin};\ Z_2)/\mathrm{Sq}^1\, H^{4k+1}(B\,\mathrm{Spin};\ Z_2)$$

monomorphically into $H^{4k+3}(B\,\mathrm{Spin};\ Z_2)$.

For any closed Spin manifold $M^{8k+2}$ and class $w \in H^{4k-1}(M;\ Z_2)$ one has

$$\begin{aligned}\tau^*(\mathrm{Sq}^1\theta)w[M] &= \mathrm{Sq}^1\tau^*(\theta)w[M]\\ &= \big(v_1\tau^*(\theta)w + \tau^*(\theta)\,\mathrm{Sq}^1 w\big)[M] = \tau^*(\theta)\rho\beta w[M]\end{aligned}$$

where $\beta$: $H^{4k-1}(M;\ Z_2) \to H^{4k}(M;\ Z)$ is the Bockstein. Then

$$\begin{aligned}\tau^*(\mathrm{Sq}^1\theta)w[M] &= \rho\beta w \mathrm{Sq}^2\rho\beta w[M] = \mathrm{Sq}^1 w \cdot \mathrm{Sq}^2\mathrm{Sq}^1 w[M]\\ &= \big(v_1 w \mathrm{Sq}^2\mathrm{Sq}^1 w + w \cdot \mathrm{Sq}^1\mathrm{Sq}^2\mathrm{Sq}^1 w\big)[M] = w \cdot \mathrm{Sq}^2\mathrm{Sq}^2 w[M]\\ &= \big(v_2 \cdot w\mathrm{Sq}^2 w + \mathrm{Sq}^2 w \cdot \mathrm{Sq}^2 w + \mathrm{Sq}^1 w \cdot \mathrm{Sq}^1\mathrm{Sq}^2 w\big)[M]\\ &= \big(v_{4k+1}\mathrm{Sq}^2 w + v_1\big(w\mathrm{Sq}^1\mathrm{Sq}^2 w\big)\big)[M]\end{aligned}$$

and $v_1 = 0 = v_{4k+1}$ in $M$, so this is zero. By Poincaré duality, this gives $\tau^*(\mathrm{Sq}^1\theta) = 0$. $\square$

*Note*. Because $H^7(B\,\mathrm{Spin};\ Z_2) = Z_2$, for $k = 1$ one has $\mathrm{Sq}^1\theta = w_7$, and has Wilson's result [**W**] that $w_7$ is zero in every 10 dimensional Spin manifold. Also $w_7 = \mathrm{Sq}^3 v_4$ and $\theta = \mathrm{Sq}^2 v_4 \in H^6(B\,\mathrm{Spin};\ Z_2) = Z_2$.

**4. A calculation.** One now turns attention to the cofibration (for $k \geqslant 2$)

$$\Sigma^{r-4k}K(Z_2, 4k) \xrightarrow{g} K(Z_2, r) \xrightarrow{h} Y_r$$

with $r$ large, and may write down $\tilde{H}^*(Y_r;\ Z_2)$. The kernel of $g^*$ has a basis given by the classes $\mathrm{Sq}^I i_r$ with $I$ admissible and having excess greater than $4k$, and writing $\sigma$ for $\sigma^{r-4k}$, $i$ for $i_{4k}$, the kernel of $h^*$ or image of $\delta$ has a basis given by classes $\delta\sigma\mathrm{Sq}^{I_1}i \cdots \mathrm{Sq}^{I_s}i$ for which the $I_j$ are admissible, have excess less than $4k$, and for which $s > 1$ and $(I_1, \ldots, I_s) \neq (J, \ldots, J)$ with $2^t$ $J$'s, $t > 0$; i.e., not the $2^t$th power of an indecomposable.

In order to study $\tilde{H}^*(M\mathrm{Spin}_{8s} \wedge Y_r;\, Z_2)$, one recalls that $\tilde{H}^*(M\mathrm{Spin}_{8s};\, Z_2)$ is a free $\mathscr{A}/\mathscr{A}\,\mathrm{Sq}^1 + \mathscr{A}\,\mathrm{Sq}^2$ module on $U$ and $w_4^2U$ in dimensions $8s$ and $8s+8$ with additional generators in dimension $8s+10$ and higher. Here $s$ is to be large.

Because $\tilde{\Omega}_*^{\mathrm{Spin}}(K(Z_2, 4k))$ and $H_*(B\mathrm{Spin};\, Z_2)$ are purely 2-primary, so is $\pi_*(M\mathrm{Spin}_{8s} \wedge Y_r)$. If one then examines the Bockstein spectral sequence for $\tilde{H}^*(M\mathrm{Spin}_{8s} \wedge Y_r;\, Z_2)$ (see [**B2**]), then

$$E_1 = \tilde{H}^*(M\mathrm{Spin}_{8s} \wedge Y_r;\, Z_2), \qquad d_1 = \mathrm{Sq}^1$$

and $E^\infty$ is zero since $\tilde{H}^*(M\mathrm{Spin}_{8s} \wedge Y_r;\, Z)$ consists entirely of torsion. Thus all classes in $\ker \mathrm{Sq}^1/\mathrm{im}\,\mathrm{Sq}^1$ are related by higher order Bocksteins.

One may begin by finding a map

$$M\mathrm{Spin}_{8s} \wedge Y_r \xrightarrow{f_1} K(Z_{2^t}, 8s + r + 4k + 1)$$

for which

$$f_1^*(i_{8s+r+4k+1}) = U\{\mathrm{Sq}^{4k+1} i_r\},$$

where $\{x\}$ denotes some class with $h^*\{x\} = x$, and for which $f_1^*(\beta i_{8s+r+4k+1})$, $\beta$ being the Bockstein operation, is a nonzero class in the kernel of $\mathrm{Sq}^1$. Of course, if $t = 1$, $\beta = \mathrm{Sq}^1$. Since $\mathrm{Sq}^1\mathrm{Sq}^{4k+2} i_r = \mathrm{Sq}^{4k+3} i_r \neq 0$, one must have $f_1^*(\beta i_{8s+r+4k+1}) = U\delta\sigma i\,\mathrm{Sq}^1 i$.

LEMMA 4.1. $t = 1$.

PROOF. Clearly

$$Z_{2^t} = \pi_{8s+r+4k+1}(M\mathrm{Spin}_{8s} \wedge Y_r) \cong \pi_{8s+r+4k+1}(S^{8s} \wedge Y_r)$$

is the bottom stable homotopy group. Applying stable homotopy to the cofibration gives an exact sequence

$$\begin{array}{ccccccc} 0 & \to & \pi_{8s+r+4k+1}(S^{8s} \wedge Y_r) & \to & \pi_{8s+r+4k}(S^{8s} \wedge \Sigma^{r-4k} K(Z_2, 4k)) & \to & 0 \\ & & \| & & \| & & \\ & & Z_{2^t} & & \pi_{8k}^S(K(Z_2, 4k)) & & \end{array}$$

and according to Brown [**B3**, Lemma (1.2)], the stable homotopy group of $K(Z_2, 4k)$ is $Z_2$. $\square$

Because $M\mathrm{Spin}_{8s}$ is a product (corresponding to the decomposition of cohomology) there is also a map

$$M\mathrm{Spin}_{8s} \wedge Y_r \xrightarrow{\tilde{f}_1} K(Z_2, 8s + r + 4k + 9)$$

for which

$$\tilde{f}_1^*(i_{8s+r+4k+9}) = w_4^2U\{\mathrm{Sq}^{4k+1} i_r\}$$

and

$$f_1^*(\mathrm{Sq}^1 i_{8s+r+4k+9}) = w_4^2U\delta\sigma i\mathrm{Sq}^1 i.$$

*Note.* This is the only class in the range up to dimension $8s + r + 4k + 9$ involving the generator $w_4^2U$.

One then has $h^*f_1^*$ sending $\mathrm{Sq}^2 i_{8s+r+4k+1}$ to $U\mathrm{Sq}^{4k+2}\mathrm{Sq}^1 i_r$, $\mathrm{Sq}^3 i_{8s+r+4k+1}$ to $U\mathrm{Sq}^{4k+3}\mathrm{Sq}^1 i_r$ and $\mathrm{Sq}^2\mathrm{Sq}^3 i_{8s+r+4k+1}$ to $U\mathrm{Sq}^{4k+5}\mathrm{Sq}^1 i_r$. Also, under $f_1^*$ $\mathrm{Sq}^2\mathrm{Sq}^1 i_{8s+r+4k+1}$ goes to $U\delta\sigma i\mathrm{Sq}^2\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^2 i$, $\mathrm{Sq}^3\mathrm{Sq}^1 i_{8s+r+4k+1}$ goes to

$$U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^2 i + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^2\mathrm{Sq}^1 i + U\delta\sigma i\mathrm{Sq}^3 i\mathrm{Sq}^1 i,$$

and $\mathrm{Sq}^2\mathrm{Sq}^3\mathrm{Sq}^1 i_{8s+r+4k+1}$ goes to

$$\begin{aligned} &U\delta\sigma i\mathrm{Sq}^5\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^3 i\mathrm{Sq}^2\mathrm{Sq}^1 i \\ &\qquad + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^5 i + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^4\mathrm{Sq}^1 i. \end{aligned}$$

Because the action of $\mathscr{A}$ on $U$ gives a free $\mathscr{A}/\mathscr{A}\mathrm{Sq}^1 + \mathscr{A}\mathrm{Sq}^2$ module, one then sees that $f_1^*$ is monic.

One may now find maps $f_2$: $M\mathrm{Spin}_{8s} \wedge Y_r \to K(Z_2, 8s + r + 4k + 2)$ and $f_3$: $M\mathrm{Spin}_{8s} \wedge Y_r \to K(Z_2, 8s + r + 4k + 3)$ for which $f_2^*(i_{8s+r+4k+2}) = U\{\mathrm{Sq}^{4k+2} i_r\}$, where $\{\mathrm{Sq}^{4k+2} i_r\}$ is some class mapping to $\mathrm{Sq}^{4k+2} i_r$ under $h^*$ and $f_3^*(i_{8s+r+4k+3}) = U\delta\sigma i\mathrm{Sq}^2 i$.

Now

$$h^*f_2^*\left(\mathrm{Sq}^1 i_{8s+r+4k+2}\right) = U\mathrm{Sq}^{4k+3} i_r$$

and

$$h^*f_2^*\left(\mathrm{Sq}^2 i_{8s+r+4k+2}\right) = U\mathrm{Sq}^{4k+3}\mathrm{Sq}^1 i_r = h^*f_1^*\left(\mathrm{Sq}^3 i_{8s+r+4k+1}\right).$$

Thus

$$\begin{aligned} &f_1^*\left(\mathrm{Sq}^3 i_{8s+r+4k+1}\right) + f_2^*\left(\mathrm{Sq}^2 i_{8s+r+4k+2}\right) \\ &\qquad = \lambda U\delta\sigma i\mathrm{Sq}^3 i + \mu U\delta\sigma i\mathrm{Sq}^2\mathrm{Sq}^1 i + \nu U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^2 i \end{aligned}$$

for some $\lambda, \mu, \nu \in Z_2$. One now applies $\mathrm{Sq}^3$ to this relation, using the fact that $\mathrm{Sq}^3\mathrm{Sq}^2 = 0$ to obtain

$$\begin{aligned} &U\delta\sigma i\mathrm{Sq}^5\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^3 i\mathrm{Sq}^2\mathrm{Sq}^1 i \\ &\qquad + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^5 i + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^4\mathrm{Sq}^1 i \\ &\qquad = \lambda\left(U\delta\sigma\mathrm{Sq}^1 i\left(\mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1\right)i + U\delta\sigma i\mathrm{Sq}^5\mathrm{Sq}^1 i\right) \\ &\qquad + \mu\left(U\delta\sigma\mathrm{Sq}^3 i\mathrm{Sq}^2\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3\mathrm{Sq}^1 i\right) \\ &\qquad + \nu\left(U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^2\mathrm{Sq}^1 i\mathrm{Sq}^3 i\right) \end{aligned}$$

so $\lambda = 1 = \mu + \nu$. One also has

$$\begin{aligned} &af_1^*\left(\mathrm{Sq}^2\mathrm{Sq}^1 i_{8s+r+4k+1}\right) + bf_3^*\left(\mathrm{Sq}^1 i_{8s+r+4k+3}\right) \\ &\qquad = bU\delta\sigma i\mathrm{Sq}^3 i + aU\delta\sigma i\mathrm{Sq}^2\mathrm{Sq}^1 i + (a + b)U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^2 i \end{aligned}$$

so that proper choice of $a$ and $b$ gives all possible $\lambda$, $\mu$, $\nu$ with $\lambda + \mu + \nu = 0$. Thus, one has a relation

$$(*) \qquad \begin{aligned} &f_1^*\left(\mathrm{Sq}^3 i_{8s+r+4k+1} + \mu\mathrm{Sq}^2\mathrm{Sq}^1 i_{8s+r+4k+1}\right) + f_2^*\left(\mathrm{Sq}^2 i_{8s+r+4k+2}\right) \\ &\qquad + f_3^*\left(\mathrm{Sq}^1 i_{8s+r+4k+3}\right) = 0. \end{aligned}$$

For convenience, one lets

$$\xi = \mathrm{Sq}^3 i_{8s+r+4k+1} + \mu \mathrm{Sq}^2\mathrm{Sq}^1 i_{8s+r+4k+1} + \mathrm{Sq}^2 i_{8s+r+4k+2} + \mathrm{Sq}^1 i_{8s+r+4k+3}$$

in the cohomology of the product of Eilenberg-Mac Lane spaces. One now continues to describe the homomorphism. Applying $h^*f_2^*$ to $\mathrm{Sq}^2\mathrm{Sq}^1 i_{8s+r+4k+2}$ gives $U\cdot \mathrm{Sq}^{4k+5} i_r + U\cdot \mathrm{Sq}^{4k+4}\mathrm{Sq}^1 i_r$, and all other operations $\gamma i_{8s+r+4k+2}$ with $\gamma \in \mathscr{A}_1$ actually lie in $\mathscr{A}_1\mathrm{Sq}^2$, so that

$$\begin{aligned} \xi &= \mathrm{Sq}^2 i_{8s+r+4k+2} + \cdots, \\ \mathrm{Sq}^1 \xi &= \mathrm{Sq}^3 i_{8s+r+4k+2} + \cdots, \\ \mathrm{Sq}^2 \xi &= \mathrm{Sq}^3\mathrm{Sq}^1 i_{8s+r+4k+2} + \cdots, \\ \mathrm{Sq}^2\mathrm{Sq}^1 \xi &= (\mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1) i_{8s+r+4k+2} + \cdots, \\ \mathrm{Sq}^3\mathrm{Sq}^1 \xi &= \mathrm{Sq}^5\mathrm{Sq}^1 i_{8s+r+4k+2} + \cdots. \end{aligned}$$

Applying $f_3^*$ to $\mathrm{Sq}^1 i_{8s+r+4k+3}$ gives $U\delta\sigma i\mathrm{Sq}^3 i + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^2 i$, a fact used above without mention, $\mathrm{Sq}^2 i_{8s+r+4k+3}$ gives $U\delta\sigma i\mathrm{Sq}^3\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^3 i$, $\mathrm{Sq}^3 i_{8s+r+4k+3}$ gives $U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^3\mathrm{Sq}^1 i$, $\mathrm{Sq}^2\mathrm{Sq}^1 i_{8s+r+4k+3}$ gives

$$\begin{aligned} &U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3 i + U\delta\sigma i\mathrm{Sq}^5 i + U\delta\sigma i\mathrm{Sq}^4\mathrm{Sq}^1 i \\ &\quad + U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^2\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^3\mathrm{Sq}^1 i, \end{aligned}$$

and $\mathrm{Sq}^2\mathrm{Sq}^3 i_{8s+r+4k+3} = (\mathrm{Sq}^5 + \mathrm{Sq}^4\mathrm{Sq}^1) i_{8s+r+4k+3}$ gives $U\delta\sigma\mathrm{Sq}^2\mathrm{Sq}^1 i\mathrm{Sq}^3\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^5\mathrm{Sq}^1 i$. Finally, $\mathrm{Sq}^3\mathrm{Sq}^1 i_{8s+r+4k+3}$ goes to

$$\begin{aligned} &U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3\mathrm{Sq}^1 i + U\delta\sigma\mathrm{Sq}^2\mathrm{Sq}^1 i\mathrm{Sq}^3 i + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^5 i \\ &\quad + U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^4\mathrm{Sq}^1 i + U\delta\sigma i\mathrm{Sq}^5\mathrm{Sq}^1 i \\ &\quad = f_1^*(\mathrm{Sq}^2\mathrm{Sq}^3 \beta i_{8s+r+4k+1}) \end{aligned}$$

and $\mathrm{Sq}^5\mathrm{Sq}^1 i_{8s+r+4k+3}$ goes to zero.

One then notices that

$$\mathrm{Sq}^3 \xi = \mathrm{Sq}^5\mathrm{Sq}^1 i_{8s+r+4k+1} + \mathrm{Sq}^3\mathrm{Sq}^1 i_{8s+r+4k+3}$$

and

$$\mathrm{Sq}^2\mathrm{Sq}^3 \xi = \mathrm{Sq}^5\mathrm{Sq}^1 i_{8s+r+4k+3}$$

giving the two relations which just occurred. One then observes that the map

$$\begin{aligned} M\mathrm{Spin}_{8s} \wedge Y_r \xrightarrow{f_1\times f_2\times f_3} &K(Z_2, 8s+r+4k+1) \times K(Z_2, 8s+r+4k+2) \\ &\times K(Z_2, 8s+r+4k+3) \end{aligned}$$

has kernel in mod 2 cohomology generated over $\mathscr{A}$ by $\xi$.

One now has a map

$$f_4 \times f_4'\colon M\mathrm{Spin}_{8s} \wedge Y_r \to K(Z_2, 8s+r+4k+4) \times K(Z_2, 8s+r+4k+4)$$

with $f_4^*(i_{8s+r+4k+4}) = U\{\mathrm{Sq}^{4k+4} i_r\}$ and $f_4'^*(i'_{8s+r+4k+4}) = U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^2 i$ so that

$$h^*f_4^*(\mathrm{Sq}^1 i_{8s+r+4k+4}) = U\mathrm{Sq}^{4k+5} i_r$$

and

$$f_4'^*\left(\mathrm{Sq}^1 i'_{8s+r+4k+4}\right) = U\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^3 i.$$

This brings one to dimension $8s + r + 4k + 5$ in which questionable behavior occurs. No class described so far hits $U \cdot \mathrm{Sq}^{4k+3}\mathrm{Sq}^2 i_r$ in $M\mathrm{Spin}_{8s} \wedge K(Z_2, r)$ and $\mathrm{Sq}^1(U \cdot \mathrm{Sq}^{4k+3}\mathrm{Sq}^2 i_r) = 0$. One may choose a class $\{\mathrm{Sq}^{4k+3}\mathrm{Sq}^2 i_r\} = x$ and $\mathrm{Sq}^1 x$ will lie in the image of $\delta$, and also in the kernel of $\mathrm{Sq}^1$. Thus $\mathrm{Sq}^1 x$ is a linear combination of

$$\delta\sigma i\mathrm{Sq}^5 i + \delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^4 i = \mathrm{Sq}^1\left(\delta\sigma i\mathrm{Sq}^4 i\right),$$

$$\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^3\mathrm{Sq}^1 i = \mathrm{Sq}^1\left(\delta\sigma i\mathrm{Sq}^3\mathrm{Sq}^1 i\right),$$

and $\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3 i$. By changing $x$ to some $x + a\delta\sigma i\mathrm{Sq}^4 i + b\delta\sigma i\mathrm{Sq}^3\mathrm{Sq}^1 i$, one may assume that $\mathrm{Sq}^1 x = c\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3 i$. If $c \neq 0$, one may let $f_5$: $M\mathrm{Spin}_{8s} \wedge Y \to K(Z_2, 8s + r + 4k + 5)$ with $f_5^*(i_{8s+r+4k+5}) = U \cdot x$ and then $f_5^*(\mathrm{Sq}^1 i_{8s+r+4k+5}) = U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3 i$. If $c = 0$, then $x$ represents a nonzero class in $\ker\mathrm{Sq}^1/\mathrm{im}\,\mathrm{Sq}^1$. There is then a higher-order Bockstein $\beta$ defined on $x$ so that $\beta x$ represents a nonzero class in $(\ker\mathrm{Sq}^1/\mathrm{im}\,\mathrm{Sq}^1)_{r+4k+6}$. Because $\mathrm{Sq}^{4k+5}\mathrm{Sq}^1 i_r = \mathrm{Sq}^1\mathrm{Sq}^{4k+4}\mathrm{Sq}^1 i_r$, $\mathrm{Sq}^1\mathrm{Sq}^{4k+6} i_r = \mathrm{Sq}^{4k+7} i_r$ and $\mathrm{Sq}^1\mathrm{Sq}^{4k+4}\mathrm{Sq}^2 i_r = \mathrm{Sq}^{4k+5}\mathrm{Sq}^2 i_r$, and the facts on $\mathrm{Sq}^1$ for the image of $\delta$, this group is $Z_2$ with generator $\delta\sigma\mathrm{Sq}^1 i\mathrm{Sq}^3 i$. Since $U$ is an integral class, one can find a map $f_5$: $M\mathrm{Spin}_{8s} \wedge Y_r \to K(Z_2 v, 8s + r + 4k + 5)$ for which $f_5^*(i_{8s+r+4k+5}) = U \cdot x$ for which $f_5^*(\beta i_{8s+r+4k+5}) = U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3 i$ modulo the image of $\mathrm{Sq}^1$. By allowing the possibility that $v = 1$, one may use this description to cover the $c \neq 0$ case as well, giving a map

$$f_5\colon M\mathrm{Spin}_{8s} \wedge Y_r \to K(Z_2 v, 8s + r + 4k + 5)$$

with $f_5^*(i_{8s+r+4k+5}) = U \cdot \{\mathrm{Sq}^{4k+4} i_r\}$ and $f_5^*(\beta i_{8s+r+4k+5}) = U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^3 i$ modulo an appropriate term.

One also has a map $f_5'$: $M\mathrm{Spin}_{8s} \wedge Y_r \to K(Z_2, 8s + r + 4k + 5)$ for which $f_5'^*(i'_{8s+r+4k+5}) = U\delta\sigma i\mathrm{Sq}^4 i$. Similarly, in higher dimensions one can find maps into Eilenberg-Mac Lane spaces $K(Z_2, 8s + r + 4k + i)$ for which

$$\begin{aligned}
i = 6\colon\quad & f_6^*\left(i_{8s+r+4k+6}\right) = U\left\{\mathrm{Sq}^{4k+4}\mathrm{Sq}^2 i_r\right\},\\
& f_6'^*\left(i'_{8s+r+4k+6}\right) = U\delta\sigma i\mathrm{Sq}^5 i,\\
i = 7\colon\quad & f_7^*\left(i_{8s+r+4k+7}\right) = U\left\{\mathrm{Sq}^{4k+4}\mathrm{Sq}^2\mathrm{Sq}^1 i_r\right\},\\
& f_7'^*\left(i'_{8s+r+4k+7}\right) = U\delta\sigma i\mathrm{Sq}^6 i,\\
& f_7''^*\left(i''_{8s+r+4k+7}\right) = U\delta\sigma i\mathrm{Sq}^5\mathrm{Sq}^1 i,\\
& f_7'''^*\left(i'''_{8s+r+4k+7}\right) = U\delta\sigma i\mathrm{Sq}^4\mathrm{Sq}^2 i,\\
i = 8\colon\quad & f_8^*\left(i_{8s+r+4k+8}\right) = U\left\{\mathrm{Sq}^{4k+8} i_r\right\},\\
& f_8^{(j)*}\left(i^{(j)}_{8s+r+4k+8}\right) = \begin{cases} U\delta\sigma i\mathrm{Sq}^7 i, & j = 1,\\ U\delta\sigma\mathrm{Sq}^2 i\mathrm{Sq}^5 i, & j = 2,\\ U\delta\sigma i\mathrm{Sq}^5\mathrm{Sq}^2 i, & j = 3,\\ U\delta\sigma i\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 i, & j = 4.\end{cases}
\end{aligned}$$

By tedious and unpleasant calculation, one may then verify that the product of all of these maps

$$f\colon M\mathrm{Spin}_{8s} \wedge Y_r \to \prod_{i=1}^{8} K(G_i, 8s + r + 4k + i)$$

where

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $G_i$ | $Z_2$ | $Z_2$ | $Z_2$ | $2Z_2$ | $Z_2v + Z_2$ | $2Z_2$ | $4Z_2$ | $5Z_2$ |

induces an epimorphism in mod 2 cohomology through dimension $8s + r + 4k + 8$, and that through dimension $8s + r + 4k + 9$ the kernel is generated over $\mathscr{A}$ by $\xi$. One may then choose a minimal set of additional generators in dimension $8s + r + 4k + 9$, giving

$$\hat{f}\colon M\mathrm{Spin}_{8s} \wedge Y_r \to \prod_{i=1}^{9} K(G_i, 8s + r + 4k + i)$$

so that $\hat{f}^*$ is epic through dimension $8s + r + 4k + 9$, and has kernel generated by $\xi$ over $\mathscr{A}$ through this dimension.

Letting $F$ be the fiber of $\hat{f}$, one then has a fibration

$$F \to M\mathrm{Spin}_{8s} \wedge Y_r \xrightarrow{\hat{f}} \prod_{i=1}^{9} K(G_i, 8s + r + 4k + i)$$

and may calculate

$$\begin{aligned}\tilde{H}^*(F;\, Z_2) \cong{}& \mathscr{A}/\mathscr{A}\mathrm{Sq}^5\mathrm{Sq}^1\, j_{8s+r+4k+3}\\ &+ \text{terms of dimension } 8s + r + 4k + 9 \text{ or higher}\end{aligned}$$

where $j_{8s+r+4k+3}$ transgresses to $\xi$. The map $e\colon F \to K(Z_2, 8s + r + 4k + 3)$ with $e^*(i_{8s+r+4k+3}) = j_{8s+r+4k+3}$ induces an isomorphism in mod 2 cohomology in dimension less than or equal to $8s + r + 4k + 8$. Thus $e$ induces an isomorphism in homotopy through dimension $8s + r + 4k + 7$ and is epic in dimension $8s + r + 4k + 8$ (which is obvious).

One may now read off the homotopy groups to obtain

LEMMA 4.2. *For $j = 4k$ with $k \geqslant 1$,*

$$\pi_{r+4k+7}(M\mathrm{Spin} \wedge Y_r) = Z_2 + Z_2 + Z_2 + Z_2$$

*with the nonzero classes being detected by $U\{\mathrm{Sq}^{4k+4}\mathrm{Sq}^2\mathrm{Sq}^1\, i_r\}$, $U\delta\sigma i\mathrm{Sq}^6\, i$, $U\delta\sigma i\mathrm{Sq}^5\mathrm{Sq}^1\, i$, and $U\delta\sigma i\mathrm{Sq}^4\mathrm{Sq}^2\, i$. In addition, there is a class in $\pi_{r+4k+3}(M\mathrm{Spin} \wedge Y_r)$ which is detected by $U\delta\sigma i\mathrm{Sq}^2\, i$.*

*Note.* The class in $\pi_{r+4k+3}(M\mathrm{Spin} \wedge Y_r)$ also occurs for $k = 1$, since for $k = 1$, the description of $\tilde{H}^*(Y_r, Z_2)$ is correct through dimension $r + 4k + 5$, the first problem being the class $\delta\sigma i\mathrm{Sq}^5\, i$.

PROOF. One has

$$\begin{array}{ccccccc} \pi_{8s+r+4k+7}(F) & \to & \pi_{8s+r+4k+7}(M\mathrm{Spin}_{8s} \wedge Y_r) & \to & G_7 & \to & \pi_{8s+r+4k+6}(F) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

and

$$\begin{array}{ccccc} \pi_{8s+r+4k+3}(M\mathrm{Spin}_{8s} \wedge Y_r) & \to & G_3 & \to & \pi_{8s+r+4k+2}(F). \quad \square \\ & & & & \| \\ & & & & 0 \end{array}$$

**5. The main results.** Having done all the hard work, one can now obtain

PROPOSITION 5.1. *For a closed Spin manifold $M^{8k+2}$ of dimension $8k+2$ and class $z \in H^{4k}(M; Z)$, $\rho z \mathrm{Sq}^2 \rho z[M] = \rho z \mathrm{Sq}^2 v_{4k}[M]$.*

PROOF. For $k = 1$, $\theta = \mathrm{Sq}^2 v_4$ is the only nonzero class in $H^6(B\mathrm{Spin}; Z_2)$. Assuming $k \geqslant 3$, $\mathrm{Sq}^1\theta \in H^{4k+3}(B\mathrm{Spin}; Z_2)$ is zero in every Spin manifold of dimension $8k+2$ and hence in every Spin manifold of smaller dimension. If one considers the sequence

$$\tilde{\Omega}^{\mathrm{Spin}}_{8k-1}(K(Z_2, 4k-4)) \xrightarrow{g} H_{4k+3}(B\mathrm{Spin}; Z_2) \xrightarrow{h} \pi_{r+4(k-1)+7}(M\mathrm{Spin} \wedge Y_r)$$
$$\xrightarrow{\partial} \tilde{\Omega}^{\mathrm{Spin}}_{8k-2}(K(Z_2, 4k-4)),$$

then $k - 1 \geqslant 2$ and $\pi_{r+4(k-1)+7}(M\mathrm{Spin} \wedge Y_r) = 4Z_2$. The classes detected by $U\delta\sigma i \mathrm{Sq}^6 i$, $U\delta\sigma i \mathrm{Sq}^5 i$, and $U\delta\sigma i \mathrm{Sq}^4\mathrm{Sq}^2 i$ map nontrivially under $\partial$, i.e. the value of $U\delta\sigma y$ on $a$ is the value of $y$ on $\partial a$. Thus, the image of $h$ or cokernel of $g$ is at most $Z_2$ and is detected by $U\{\mathrm{Sq}^{4k}\mathrm{Sq}^2\mathrm{Sq}^1 i\}$. Letting $N^{r+4k+3}$ be a Spin manifold with $w \in H^{4k+3}(N; Z_2)$ to realize a class in $\tilde{\Omega}^{\mathrm{Spin}}_{r+4k+3}(K(Z_2, r)) \cong H_{4k+3}(B\mathrm{Spin}; Z_2)$, the value of $U\{\mathrm{Sq}^{4k}\mathrm{Sq}^2\mathrm{Sq}^1 i\}$ on $(N, w)$ is

$$\begin{aligned} \mathrm{Sq}^{4k}\mathrm{Sq}^2\mathrm{Sq}^1 w[N] &= v_{4k}\mathrm{Sq}^2\mathrm{Sq}^1 w[N] = \{v_2 v_{4k}\mathrm{Sq}^1 w + \mathrm{Sq}^2 v_{4k}\mathrm{Sq}^1 w\}[N] \\ &= \{v_1 \mathrm{Sq}^2 v_{4k} w + \mathrm{Sq}^1\mathrm{Sq}^2 v_{4k} \cdot w\}[N] = \{\mathrm{Sq}^3 v_{4k} \cdot w\}[N]. \end{aligned}$$

Thus, the only class in $H^{4k+3}(B\mathrm{Spin}; Z_2)$ which can vanish on the image of $g$ is $\mathrm{Sq}^3 v_{4k}$. Thus $\mathrm{Sq}^1\theta = \mathrm{Sq}^3 v_{4k} = \mathrm{Sq}^1\mathrm{Sq}^2 v_{4k}$ and $\theta = \mathrm{Sq}^2 v_{4k}$.

Finally, for the case $k = 2$, one could presumably redo all of the calculations of the previous section for the case $k = 1$. However, being given $M^{18}$ and a class $z \in H^8(M; Z)$ with Wu class $v(M) = 1 + v_4' + v_8'$ one can let $u \in H^4(HP^2; Z)$ and consider $u \otimes z \in H^{12}(HP^2 \times M; Z)$ so that

$$\begin{aligned} \rho z \mathrm{Sq}^2 \rho z[M] &= \rho(u \otimes z)\, \mathrm{Sq}^2 \rho(u \otimes z)[HP^2 \times M] \\ &= \rho(u \otimes z)\, \mathrm{Sq}^2 v_{12}[HP^2 \times M] \\ &= \rho(u \otimes z)\, \mathrm{Sq}^2(\rho u \otimes v_8')[HP^2 \times M] \\ &= \rho z \mathrm{Sq}^2 v_8'[M]. \end{aligned}$$

Thus, the result for $k = 3$ implies it for $k = 2$. $\square$

COROLLARY 5.2 [**W**]. $\mathrm{Sq}^3 v_{4k} = 1\ \mathrm{Sq}^{4k}\mathrm{Sq}^2\mathrm{Sq}^1$ *is zero in every closed Spin manifold of dimension* $8k+2$.

PROOF. Having seen that $\theta = \mathrm{Sq}^2 v_{4k}$ gives this. $\square$

*Note.* With the exception of the case $k = 2$, one has shown that this is the *only* nonzero class of dimension $4k+3$ which is zero in every manifold of dimension $8k+2$ (or $8k-1$).

COROLLARY 5.3. *For a closed spin manifold* $M^{8k+2}$ *of dimension* $8k+2$, $w_4 w_{8k-2}[M] = v_{4k}\mathrm{Sq}^2 v_{4k}[M]$ *is the rank modulo* 2 *of the form* [ , ] *on integral cohomology.*

PROOF. Consider the form

$$[\ ,\ ]\colon H^{4k}(M;\, Z) \otimes H^{4k}(M;\, Z) \to Z_2\colon [x, y] = \rho x \mathrm{Sq}^2 \rho y[M].$$

By standard facts about forms (as in [**LMP**, §2]), there is a class $v \in H^{4k}(M;\, Z)$, well-defined modulo the annihilator of the form, for which $[x, y] = [x, x]$ for all $x$ and $[v, v]$ is the rank modulo 2 of the form [ , ]. In $H^*(B\mathrm{Spin};\, Z_2)$, it is well known [**ABP**] that $\mathrm{Sq}^1 v_{4k} = 0$, and the kernel of $\mathrm{Sq}^1$ is the image of the reduction of $H^*(B\mathrm{Spin};\, Z)$. Thus there is a class $w \in H^*(B\mathrm{Spin};\, Z)$ with $\rho w = v_{4k}$. By the proposition $\tau^*(w) \in H^{4k}(M;\, Z)$ is a suitable choice for $v$ and so the rank mod 2 of [ , ] is $[\tau^*(w), \tau(w)] = \rho\tau^*(w)\mathrm{Sq}^2\rho\tau^*(w)[M] = v_{4k}\mathrm{Sq}^2 v_{4k}[M]$. Finally,

$$\begin{aligned} v_{4k}\mathrm{Sq}^2 v_{4k}[M] &= \mathrm{Sq}^{4k}\mathrm{Sq}^2 v_{4k}[M] \\ &= \left\{\mathrm{Sq}^4\mathrm{Sq}^{4k-2}v_{4k} + \binom{4k-3}{4}\mathrm{Sq}^{4k+2}v_{4k}\right\}[M] \\ &= v_4\mathrm{Sq}^{4k-2}v_{4k}[M] \end{aligned}$$

and since $v_i(M) = 0$ for $i \not\equiv 0\ (4)$, $v_4 = w_4$ and $\mathrm{Sq}^{4k-2}v_{4k} = w_{8k-2}$ for $w = \mathrm{Sq}\, v$. $\square$

OBSERVATION. *There is no class* $y \in H^{4k+2}(B\mathrm{Spin};\, Z_2)$ *with* $k > 0$ *so that for all closed Spin manifolds* $M^{8k+2}$ *and* $x \in H^{4k}(M;\, Z_2)$ *one has*

$$x\mathrm{Sq}^2 x[M] = x\tau^*(y)[M].$$

PROOF. From the calculations in the previous section (valid for $k \geqslant 1$) one has a class $a \in \pi_{r+4k+3}(M\mathrm{Spin} \wedge Y_r)$ for which $U\delta\sigma i\mathrm{Sq}^2 i$ has a nonzero value. In the sequence

$$\pi_{r+4k+3}(M\mathrm{Spin} \wedge Y_r) \xrightarrow{\partial} \tilde{\Omega}^{\mathrm{Spin}}_{8k+2}(K(Z_2, 4k)) \to H_{4k+2}(B\mathrm{Spin};\, Z_2)$$

$\partial a$ is given by an $M^{8k+2}$ and class $x$ with $x\mathrm{Sq}^2 x[M] \neq 0$ and so that $x\tau^*(y)[M] = 0$ for all $y$. $\square$

*Note.* This shows that the restriction to integral classes was absolutely crucial.

OBSERVATION. *There is no class* $y \in H^{4k+4}(B\mathrm{Spin};\, Z_2)$ *so that for all closed Spin manifolds* $M^{8k+6}$ *and* $z \in H^{4k+2}(M, Z)$ *one has*

$$\rho z\mathrm{Sq}^2\rho z[M] = \rho z\tau^*(y)[M].$$

PROOF. Let $M^{8k+6} = HP^{2k} \times \mathbf{C}P^3$ and $z = u^k a$ where $u \in H^4(HP^{2k}; Z)$, $a \in H^2(\mathbf{C}P^3; Z)$. Then $\rho z \mathrm{Sq}^2 \rho z[M] = \rho(u^k a)\rho(u^k a^2)[M] \neq 0$. Also $w(M) = (1+\rho u)^{2k+1}(1+\rho a)^4 = (1+\rho u)^{2k+1}$ and for any $y \in H^{4k+4}(B\mathrm{Spin}; Z_2)$, $\tau^*(y) = \lambda \rho u^{k+1}$ for some $\lambda \in Z_2$. Thus $\rho z \tau^*(y)[M] = \lambda \rho u^{2k+1} \rho a[M] = 0$. $\square$

OBSERVATION. *There is no class $y \in H^{4k+1}(B\mathrm{Spin}; Z_2)$ with $k > 0$ so that for all closed Spin manifolds $M^{8k}$ and $z \in H^{4k-1}(M; Z)$ one has*

$$\rho z \mathrm{Sq}^2 \rho z[M] = \rho z \tau^*(y)[M].$$

PROOF. Let $M^{8k} = HP^{2k-2} \times G_2(R^6)$, where $G_2(R^6)$ is the Grassmannian of 2-planes in $R^6$. Then $H^*(G_2(R^6); Z_2)$ is the $Z_2$ polynomial ring on the universal Stiefel-Whitney classes $w_1$, $w_2$ modulo the relations $(1/(1+w_1+w_2))_i = 0$ if $i > 4$. One has $w(G_2(R^6)) = (1+w_1+w_2)6/(1+w_1^2)$, so that $G_2(R^6)$ is a Spin manifold, and for any $y \in H^{4k+1}(B\mathrm{Spin}; Z_2)$, $\tau^*(y) = 0$ in $M$ since all odd dimensional Stiefel-Whitney classes are zero. Let $a = \beta w_2 \in H^3(G_2(R^6); Z)$ be the integral Bockstein of $w_2$, so $\rho a = \rho\beta w_2 = \mathrm{Sq}^1 w_2 = w_1 w_2$, and let $z$ be $u^{k-1}a$. Then

$$\rho z \mathrm{Sq}^2 \rho z[M] = \mathrm{Sq}^1 w_2 \mathrm{Sq}^2 \mathrm{Sq}^1 w_2 \left[G_2(R^6)\right] \neq 0. \qquad \square$$

In dimensions $8k+4$ with $k > 0$, one may similarly consider $HP^{2k-2} \times M^{12}$ where $M^{12}$ is a Spin manifold having a class $a \in H^5(M; Z)$ with $\rho a \mathrm{Sq}^2 \rho a[M] \neq 0$, and may let $z = u^{k-1}a$ to give $\rho z \mathrm{Sq}^2 \rho z[HP^{2k-2} \times M] \neq 0$. The Wu class of $M$ has the form $1 + v_4$ ($v_i = 0$ if $i \not\equiv 0 \bmod 4$ or $i > 6$) so $w(M) = 1 + w_4 + w_6 + w_7 + w_8$ and by Wilson [**W**], $w_7 = 0$. Thus $w(HP^{2k-2} \times M)$ consists entirely of even dimensional classes, and for any $y \in H^{4k+3}(B\mathrm{Spin}; Z_2)$, $\tau^*(y) = 0$.

By calculation, one can show that $(M^{12}, a)$ exists. To exhibit such calculations would be a travesty; one would prefer a specific example.

*Note.* In dimensions $8k$ and $8k+4$, with $k = 0$, $y = 0$ would give the universal class. Similarly, $y = 0$ suffices for mod 2 cohomology in dimensions $8k+2$ with $k = 0$.

**6. A technical extension.** Having seen that the main result does not hold for arbitrary mod 2 cohomology classes, one is led to ask whether weaker conditions than being reduced integral are sufficient. One does, in fact, have

PROPOSITION 6.1. *For a closed Spin manifold $M^{8k+2}$ of dimension $8k+2$ and class $x \in H^{4k}(M; Z_2)$, one has*

$$x \mathrm{Sq}^2 x[M] = x \mathrm{Sq}^2 v_{4k}[M]$$

*if $\mathrm{Sq}^1 x = 0$, i.e. if $x$ is the reduction of a $Z_4$ class.*

COROLLARY 6.2. *For a closed Spin manifold $M^{8k+2}$ of dimension $8k+2$, $w_4 w_{8k-2}[M]$ is the rank modulo 2 of the form $[\ ,\ ]$ on $(\ker \mathrm{Sq}^1)^{4k}$ or $H^{4k}(M\colon Z_2 s)$ for any $s > 1$.*

*Note*. The results of **[LMP]** relate the form $(x, y) = x\mathrm{Sq}^1 y[M]$ to the torsion in homology in a very precise way. These results indicate that there is some relation on the torsion for Spin manifolds of dimension $8k + 2$ because the rank of the form is independent of $s$, but the relation is vague.

PROOF. One has a cofibration

$$\Sigma^{r-4k}K(Z_4, 4k) \to K(Z_4, r) \to W_r$$

giving an exact sequence

$$\pi_{r+4k+3}(M\mathrm{Spin} \wedge W_r) \xrightarrow{\partial} \tilde{\Omega}^{\mathrm{Spin}}_{8k+2}(K(Z_4, 4k)) \xrightarrow{a} H_{4k+2}(B\mathrm{Spin};\ Z_4) \to \cdots .$$

One may then analyze $\tilde{H}^*(W_r;\ Z_2)$ and find

$$\begin{array}{ll} \dim(r + 4k + 1) & [\mathrm{Sq}^{4k+1} i_r], \\ \dim(r + 4k + 2) & [\mathrm{Sq}^{4k+2} i_r],\ \delta\sigma^{r-4k} i_{4k}\beta i_{4k}, \\ \dim(r + 4k + 3) & [\mathrm{Sq}^{4k+3} i_r],\ [\mathrm{Sq}^{4k+2}\beta i_r],\ \delta\sigma^{r-4k} i_{4k}\mathrm{Sq}^2 i_{4k}, \end{array}$$

where $\beta$ denotes the Bockstein. Since $\mathrm{Sq}^2[\mathrm{Sq}^{4k+1} i_r]$ goes to $\mathrm{Sq}^{4k+2}\mathrm{Sq}^1 i_r = 0$ in $K(Z_4, r)$, one has $\mathrm{Sq}^2[\mathrm{Sq}^{4k+1} i_r] = \mu\delta\sigma^{r-4k} i_{4k}\mathrm{Sq}^2 i_{4k}$ for some $\mu \in Z_2$.

If one considers the maps $K(Z, n) \to K(Z_4, n)$, one has an induced map $X_r \xrightarrow{b} W_r$ so that $b^*$: $\tilde{H}^*(W_r;\ Z_2) \to \tilde{H}^*(X_r;\ Z_2)$ sends $[\mathrm{Sq}^{4k+1} i_r]$ to $[\mathrm{Sq}^{4k+1} i_r]$. Thus $\mathrm{Sq}^2[\mathrm{Sq}^{4k+1} i_r] \neq 0$ in $W_r$, because its image in $X_r$ is nonzero, and $\mu \neq 0$.

Thus

$$\phi\colon \tilde{\Omega}^{\mathrm{Spin}}_{8k+2}(K(Z_4, 4k)) \to Z_2\colon (M, f) \to (f^*i)\,\mathrm{Sq}^2 f^*i[M]$$

is zero on the image of $\partial$, and is given by a homomorphism $\operatorname{im} a \to Z_2$. Since all torsion in $H_*(B\mathrm{Spin};\ Z)$ is of order 2, $H_{4k+2}(B\mathrm{Spin};\ Z_4) \xrightarrow{\rho} H_{4k+2}(B\mathrm{Spin};\ Z_2)$ is monic, and there is a class $\theta \in H^{4k+2}(B\mathrm{Spin};\ Z_2)$ so that

$$x\mathrm{Sq}^2 x[M] = \tau^*(\theta) \cdot x[M]$$

for all closed Spin $M^{8k+2}$ and $x \in (\ker \mathrm{Sq}^1)^{4k} = \rho H^{4k}(M;\ Z_4)$.

One must again identify $\theta$, but this is just a repetition of the arguments. $\theta$ is well defined only modulo the image of $\mathrm{Sq}^1$, hence is determined by $\mathrm{Sq}^1\theta$, and $\mathrm{Sq}^1\theta$ is zero in all Spin $M^{8k+2}$. By uniqueness, $\theta = \mathrm{Sq}^2 v_{4k}$ mod image $\mathrm{Sq}^1$ for $k \geqslant 3$, and this implies that $\theta$ can be taken to be $\mathrm{Sq}^2 v_{4k}$ for smaller $k$. □

*Note*. The argument for $Z_4$ is really identical with that for $Z$ classes, and this presentation has simply used the $Z$ argument to give the Steenrod operations in $W_r$. The equivalence of the ranks of the forms for $Z$ and $Z_2 s$ cohomology follows from the fact that the class $v_{4k}$ is reduced integral.

*Note*. One can analyze the form [ , ] simply by knowing $H^*(M;\ Z_2)$ as algebra over the Steenrod algebra, since that gives $(\ker \mathrm{Sq}^1)^{4k}$. Working with $\rho H^{4k}(M;\ Z)$ would require extra information.

COMMENT. This extension to $Z_4$ classes was inspired by a suggestion of Steven M. Kahn. Using this extension the methods of [**K**] may be applied to prove

PROPOSITION 6.3. *If $M^{8k+2}$ is a closed Spin manifold of dimension $8k+2$ with an involution $T$ of odd type preserving the Spin structure, then*

$$w_4 w_{8k-2}[M] \equiv \chi(F^{8*}) \equiv \chi(F^{8*+4}) \pmod 2$$

*where $\chi$ is the Euler characteristic and $F^{8*+j}$ is the part of the fixed set of $T$ having dimension $j$* mod 8.

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